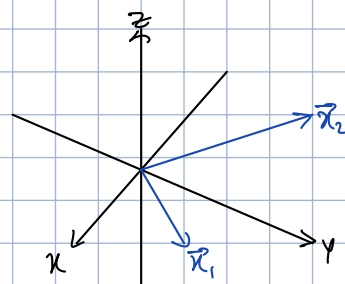## TOPIC 1: The Gram-Schmidt Process

Motivation: Suppose $\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{x}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$.
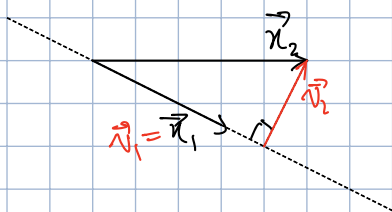

$\vec{x}_1$ and $\vec{x}_2$ are linearly independent $\Rightarrow$ they form a basis for
$$W = \text{Span}\{\vec{x}_1, \vec{x}_2\},$$
but $\vec{x}_1$ and $\vec{x}_2$ do not give an <u>orthogonal</u> basis for W
How might we construct an orthogonal basis for W?

Example: Let W be the subspace of $\mathbb{R}^3$ spanned by $\vec{x}_1$ and $\vec{x}_2$. Construct an orthogonal basis for W.

$$\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \vec{x}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$\vec{v}_2$ is a linear combination of $\vec{x}_1$ and $\vec{x}_2$.

Set $\vec{v}_1 = \vec{x}_1$, $\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{x}_1} \vec{x}_2$

$$= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{x}_1 \cdot \vec{v}_1} \vec{x}_1$$

$$= \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$\therefore$ Our orthogonal basis is $\{\vec{v}_1, \vec{v}_2\}$.

The Gram-Schmidt Process with Two Vectors

Suppose we are given a set of vectors $\{\vec{x}_1, \vec{x}_2\}$ in $\mathbb{R}^n$. We can construct an orthogonal basis, $\{\vec{v}_1, \vec{v}_2\}$ for the space that they span, W, with the following process:

$$\vec{v}_1 = \vec{x}_1$$
$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

We can show that if $\{\vec{x}_1, \vec{x}_2\}$ are independent, then $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal basis for W:
* $\vec{v}_1$ and $\vec{v}_2$ are in W.
* $\vec{v}_1$ and $\vec{v}_2$ span W.
* $\vec{v}_1$ and $\vec{v}_2$ are orthogonal: $\vec{v}_1 \cdot \vec{v}_2 = 0$.

Exercise:
Vectors $\vec{u}$ and $\vec{v}$ form a basis for W.

$$\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$$

Suppose we want to construct an orthogonal basis for W, which we will denote by the vectors $\hat{u}$ and $\hat{v}$. If we apply the Gram-Schmidt process to identify an orthogonal basis for W, we can set:

$$\hat{u} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

The other vector in our orthogonal basis is: $\quad \vec{v} - \text{proj}_{\vec{u}}\,\vec{v} =$

$$\hat{v} = \begin{pmatrix} x_1 \\ x_2 \\ -3 \end{pmatrix} \qquad\qquad \vec{v} = \vec{u}$$

Find x_1 and x_2.

$$\vec{v} - \text{proj}_{\vec{u}}\,\vec{v} = \vec{v} - \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u}$$

$$= \vec{v} - \frac{4+4-2}{1^2+2^2+1^2}\,\vec{u} = \vec{v} - \vec{u}$$

$$\vec{v} - \text{proj}_{\vec{u}}\,\vec{v} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}. \quad \therefore x_1 = 3,\ x_2 = 0.$$

Example:
The vectors below span a subspace W of $\mathbb{R}^4$. Construct an orthogonal basis for W.

$$\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{x}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

Set $\vec{v}_1 = x_1$, $\vec{w}_1 = \text{span}\{v_1\}$

$$\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{w}_1}\,\vec{x}_2$$

$$= \vec{x}_2 - \frac{\vec{x}_2\cdot\vec{x}_1}{\vec{x}_1\cdot\vec{x}_1}\,\vec{x}_1 = \vec{x}_2 - \frac{0+1+2+0}{1+1+1+0}\,\vec{x}_1$$

$$= \vec{x}_2 - \vec{x}_1$$

$$\vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Check: $\vec{v}_1\cdot\vec{v}_2 = 0$. Set $\vec{w}_2 = \text{span}\{\vec{v}_1, \vec{v}_2\}$

$$\vec{v}_3 = \vec{x}_3 - \text{proj}_{\vec{x}_2}\,\vec{w}_2$$

$$= \vec{x}_3 - \left(\frac{\vec{x}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\,\vec{v}_1 + \frac{\vec{x}_3\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\,\vec{v}_2 \right)$$

$$= \vec{x}_3 - \left(\frac{0}{3}\,\vec{v}_1 + \frac{-1}{3}\,\vec{v}_2 \right) = \vec{x}_3 + \frac{1}{3}\,\vec{v}_2$$

$$\vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/3 \\ 0 \\ 1/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 1 \\ -2/3 \\ 1/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 \\ 3 \\ -2 \\ 1 \end{pmatrix}$$

$\Rightarrow$ orthogonal basis for W is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

The Gram-Schmidt Process

Suppose $\{\vec{x}_1, \ldots, \vec{x}_p\}$ are a basis for a subspace W of $\mathbb{R}^n$.

$$W_1 = \text{Span}\{\vec{v}_1\} \qquad\qquad \vec{v}_1 = \vec{x}_1$$
$$W_2 = \text{Span}\{\vec{v}_1, \vec{v}_2\} \qquad \vec{v}_2 = \vec{x}_2 - \text{proj}_{W_1}\vec{x}_2$$
$$W_3 = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \qquad \vec{v}_3 = \vec{x}_3 - \text{proj}_{W_2}\vec{x}_3$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$W_p = \text{span}\{\vec{v}_1, \ldots, \vec{v}_p\} \qquad \vec{v}_p = \vec{x}_p - \text{proj}_{W_{p-1}}\vec{x}_p$$

Then, $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is an **orthogonal** basis for W.

The Gram-Schmidt Process can also be written this way.

$$\vec{v}_1 = \vec{x}_1$$
$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1$$
$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$$
$$\vdots$$
$$\vec{v}_p = \vec{x}_p - \frac{\vec{x}_p \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \cdots - \frac{\vec{x}_p \cdot \vec{v}_{p-1}}{\vec{v}_{p-1} \cdot \vec{v}_{p-1}}\vec{v}_{p-1} = \vec{x}_p - \sum_{i=1}^{p}\frac{\vec{x}_p \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}\vec{v}_i$$

Exercise:
The three vectors $\vec{x}_1$, $\vec{x}_2$ and $\vec{x}_3$ form a basis for subspace W, where

$$\vec{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{x}_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Suppose that an orthogonal basis for W is the set:

$$\{\hat{v}_1, \hat{v}_2, \hat{v}_3\}.$$

Note that $\vec{x}_1$ and $\vec{x}_2$ are orthogonal. Therefore, to construct an orthogonal basis for W, we can set $\hat{v}_1 = \vec{x}_1$ and $\hat{v}_2 = \vec{x}_2$. To determine $\hat{v}_3$ we may use the Gram-Schmidt process. Find $\hat{v}_3$.

The projection of $\vec{x}_3$ onto the space spanned by $\vec{x}_1$ and $\vec{x}_2$, W_2, is denoted by $\text{proj}_{W_2}\vec{x}_3$.

The difference between $\vec{x}_3$ and its projection on W_2 is in W, and is orthogonal to both $\vec{x}_1$ and $\vec{x}_2$. So it is the vector that we need to compute.

$$\hat{v}_3 = \text{proj}_{W_2}\vec{x}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{x}_1}{\vec{x}_1 \cdot \vec{x}_1}\vec{x}_1 - \frac{\vec{x}_3 \cdot \vec{x}_2}{\vec{x}_2 \cdot \vec{x}_2}\vec{x}_2$$
$$= \vec{x}_3 - \frac{2}{2}\vec{x}_1 - \frac{4}{2}\vec{x}_2 = \vec{x}_3 - \vec{x}_1 - 2\vec{x}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} *$$

TOPIC 2: The QR Factorization

Recall: **DEFINITION**
↳ A set of vectors form an orthonormal basis if the vectors are mutually orthogonal and have unit length.

Example: The two vectors below form an orthogonal basis for a subspace W. Obtain an orthonormal basis for W.

$$\vec{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \qquad \hat{v}_1 = \frac{1}{\sqrt{13}}\vec{v}_1, \quad \hat{v}_2 = \frac{1}{\sqrt{14}}\vec{v}_2$$

QR Factorization uses this concept.

QR Factorization

Theorem:
Any $m \times n$ matrix A with linearly independent columns has the QR factorization

$$A = QR$$

where:
* Q is $m \times n$, its columns are an orthonormal basis for Col A.
* R is $n \times n$, upper triangular, with positive entries on its diagonal.

Notes on QR Factorization:
* We are not considering the case when A has linearly dependent columns.
* Q can be obtained using the Gram-Schmidt process.
* To obtain R, we use $R = Q^T A$, because

$$A = QR$$
$$Q^T A = Q^T Q R$$
$$Q^T A = R \qquad\qquad \because Q^T Q = I$$

* The length of the jth column of R is equal to the length of the jth column of A.

Example: Construct the QR decomposition for $A = \begin{pmatrix} 3 & -2 \\ 2 & 3 \\ 0 & 1 \end{pmatrix}$

The columns of Q form an orthonormal basis for Col A, but the columns of A are already orthogonal (if they were not, we could use Gram-Schmidt).

Construct Q by dividing each column by its own length.

$$Q = \begin{pmatrix} 3/\sqrt{13} & -2/\sqrt{14} \\ 2/\sqrt{13} & 3/\sqrt{14} \\ 0 & 1/\sqrt{14} \end{pmatrix}$$

Next, we construct R.

$$R = Q^T A = \begin{pmatrix} 3/\sqrt{13} & 2/\sqrt{13} & 0 \\ -2/\sqrt{14} & 3/\sqrt{14} & 1/\sqrt{14} \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{13} & 0 \\ 0 & \sqrt{14} \end{pmatrix} *$$

Exercise:
Suppose A is the matrix below.

$$A = \begin{pmatrix} 2 & 2 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The columns of A are independent, so A has the QR Factorization

$$A = QR$$

where $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$. Find R.

$$R = Q^T A$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$$